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Ground states for Schrödinger-type equations with nonlocal nonlinearity

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Abstract

The problem of the existence of stable solitary wave solutions for nonlinear Schrödinger-type equations with a generalized cubic nonlinearity is considered. These types of equations have recently arisen in the context of optical communications as averaging approximations to nonlinear dispersive equations with widely separated time scales. In this paper, it is shown that under general conditions on the kernel of the nonlocal term, stable standing wave solutions exist for these equations.
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1. Introduction

Nonlinear Schrödinger-type equations with nonlocal nonlinearity have arisen as asymptotic models governing a variety of physical phenomena, ranging from the flow of a superfluid around an impurity, the propagation of electromagnetic waves in plasmas, Bose–Einstein condensation, and the slow evolution of internal waves in fluids of great depth [3–5,18,19,22]. In recent years, such equations have come up in the context of optical communications as averaging approximations to the nonlinear dispersive equations which describe pulse propagation in various types of optical fibers. In modern optical

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communication systems, the variations of an optical pulse's envelope dynamics are governed by equations of the form

$$iu_\xi + Lu + \varepsilon|u|^2 u = 0,$$

where

$$Lu = d_2(\xi)u_{tt} + id_3(\xi)u_{ttt},$$

$d_j(\xi)$ are piecewise constant functions, and ε is the ratio of the characteristic length scale of local dispersion to that of nonlinearity, a small positive parameter. Specific examples of these equations include the conventional dispersion managed nonlinear Schrödinger (DMNLS) equation, where $d_2(\xi)$ is periodic and $d_3(\xi) = 0$, the higher order dispersion managed nonlinear Schrödinger (HODMNLS) equation, where both $d_2(\xi)$ and $d_3(\xi)$ are periodic, and the "random" dispersion managed nonlinear Schrödinger equation (RDMNLS), where $d_2(\xi)$ has a stochastic component and $d_3(\xi) = 0$ [1,2,8,10,20]. Solutions of these equations evolve on two widely separated time scales, which suggests the use of asymptotic methods. To arrive at an averaged equation for (1) which describes the solution's slow dynamics, one first decomposes the dispersion coefficients into varying and average components, $d_j(\xi) = \alpha_j \tilde{d}(\xi) + \varepsilon\langle d_j\rangle$, where $\tilde{d}(\xi)$ alternates between ± 1 and for physical reasons, the effective average dispersion $\varepsilon\langle d_j\rangle$ is assumed to be small and positive. Next, one performs the change of variable $u = \mathscr{T}\{v\}$, where $\mathscr{T}$ is the solution operator for the linear dispersive equation

$$iu_\xi + \alpha_2\tilde{d}(\xi)u_{tt} + i\alpha_3\tilde{d}(\xi)u_{ttt} = 0$$

and, finally, one averages over the fast variations in the transformed equation. For each of the above-mentioned cases, the averaged equation is of the general form

$$iv_z + v_{tt} + \mathscr{N}\{v\} = 0, \tag{1}$$

where

$$\hat{\mathscr{N}}\{v\} = \int_{\mathbb{R}^4} \exp(-\mathrm{i}(k - k_1 + k_2 - k_3)t)\hat{K}(\Delta)\hat{v}_1\bar{\hat{v}}_2\hat{v}_3\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}t. \tag{2}$$

Here $\hat{}$ denotes the Fourier transform, $\hat{v}_j = \hat{v}(k_j)$, $\Delta = \sum_{j=1}^3 \alpha_j\Delta_j$, $\Delta_j = k^j - k_1^j + k_2^j - k_3^j$, and $\alpha_j \in \mathbb{R}$. The kernel function $\hat{K}$ is obtained by averaging $\exp(\mathrm{i}\sum_{j=1}^3 \tilde{D}(\xi)\alpha_j\Delta_j)$ over ξ, where $\tilde{D}(\xi) = \int_0^\xi \tilde{d}(\xi')\,\mathrm{d}\xi'$

$$\hat{K}(\Delta) = \lim_{\xi\to\infty} \frac{\int_0^\xi \exp(\mathrm{i}\sum_{j=1}^3 \tilde{D}(\xi')\alpha_j\Delta_j)\,\mathrm{d}\xi'}{\xi}.$$

In typical physical situations, $\hat{K}(\Delta)$ is real-valued and symmetric about $\Delta = 0$.

A problem of both theoretical and physical interest is the existence and stability of solitary wave solutions for (1) of the form $v(z,t)=\exp(\mathrm{i}\omega z)w(t)$, where $\lim_{|t|\to\infty} w(t)=0$. In the context of optical communications, these solutions have potential for use as bit-carriers in a high speed data stream. For both the averaged DMNLS equation and averaged HODMNLS equation, the existence of stable solitary wave solutions has been established [12,21,25], with such solutions arising as constrained minima (ground states) for a corresponding Hamiltonian. In more general cases, such as the averaged RDMNLS equations, it is at first unclear that a variational technique can be used to find such solutions. One potential technical difficulty, which arises from the fact that averaging is typically performed in the Fourier domain, is that the nonlinearity may not be easily discernible in physical variables. In this paper, it is shown that under relatively mild conditions on the kernel function that are typically met in applications, namely that K is real-valued, positive, and lies in $L^1(\mathbb{R})$, the Fourier inversion of the nonlocal nonlinearity can be described in an explicit manner, leading to an explicit description of the Hamiltonian. Moreover, using the structure of this Hamiltonian, we prove the following theorem.

Theorem 1.1. *Let K be real-valued, positive, and an element of $L^1(\mathbb{R})$. Then there exists a solution to the following constrained minimization problem*:
Minimize

$$\mathscr{H}(v)=\int_{\mathbb{R}}|v_t|^2\,\mathrm{d}t-\frac{1}{2}\int_{\mathbb{R}}K(s)\int_{\mathbb{R}}|\mathscr{L}(s)\{v\}|^4\,\mathrm{d}t\,\mathrm{d}s$$

over the set of admissible functions

$$\mathscr{A}_\lambda=\left\{v\in H^1(\mathbb{R}),\ \int_{\mathbb{R}}|v|^2=\lambda\right\}.$$

This solution corresponds to a stable standing wave solution for (1).

Finally, we note that a fast numerical algorithm, first presented in [9], can be used to calculate solutions to this resulting eigenvalue problem. This work generalizes the analytical results of [12,21,25] while complementing the numerical results in [2,9].

2. Fourier inversion of the nonlinearity

The first step in our analysis is to invert the Fourier transform of the nonlinear operator. First we make the substitution

$$\hat{K}(\Delta)=\int_{\mathbb{R}}\exp(-\mathrm{i}\Delta s)K(s)\,\mathrm{d}s$$

in (2) to obtain

$$\begin{aligned}
\hat{\mathcal{N}}\{v\} &= \int_{\mathbb{R}^4} \exp(-\mathrm{i}(k-k_1+k_2-k_3)t) \\
&\quad \times \left(\int_{\mathbb{R}} \exp(-\mathrm{i}\Delta s)K(s)\,\mathrm{d}s\right)\hat{v}_1\bar{\hat{v}}_2\hat{v}_3\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}t \\
&= \int_{\mathbb{R}^4} \exp(-\mathrm{i}(k-k_1+k_2-k_3)t) \\
&\quad \times \left(\int_{\mathbb{R}} \exp\left(-\mathrm{i}\left(\sum_{j=1}^{3}\alpha_j\Delta_j\right)s\right)K(s)\,\mathrm{d}s\right)\hat{v}_1\bar{\hat{v}}_2\hat{v}_3\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \exp(-\mathrm{i}kt)\int_{\mathbb{R}} K(s)\,\exp(-\mathrm{i}(\alpha_1 k+\alpha_2 k^2+\alpha_3 k^3)s) \\
&\quad \times \left(\int_{\mathbb{R}} \exp(\mathrm{i}k_1 t)\exp(\mathrm{i}(\alpha_1 k_1+\alpha_2 k_1^2+\alpha_3 k_1^3)s)\hat{v}_1\,\mathrm{d}k_1\right. \\
&\quad \times \int_{\mathbb{R}} \exp(-\mathrm{i}k_2 t)\,\exp(-\mathrm{i}(\alpha_1 k_2+\alpha_2 k_2^2+\alpha_3 k_2^3)s)\bar{\hat{v}}_2\,\mathrm{d}k_2 \\
&\quad \left.\times \int_{\mathbb{R}} \exp(\mathrm{i}k_3 t)\,\exp(\mathrm{i}(\alpha_1 k_3+\alpha_2 k_3^2+\alpha_3 k_3^3)s)\hat{v}_3\,\mathrm{d}k_3\right)\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

We can simplify the above operator by recognizing that $\exp(\mathrm{i}(\alpha_1 k_j+\alpha_2 k_j^2+\alpha_3 k_j^3)s)\hat{v}_j$ is the Fourier transform of the solution to the linear equation

$$i v_s + i\alpha_1 v_t - \alpha_2 v_{tt} - i\alpha_3 v_{ttt} = 0. \tag{3}$$

If we denote the solution operator to this equation as $\mathcal{L}$, $\hat{\mathcal{N}}$ simplifies to

$$\begin{aligned}
\hat{\mathcal{N}}\{v\} &= \int_{\mathbb{R}} \exp(-\mathrm{i}kt)\left(\int_{\mathbb{R}} K(s)\,\exp(-\mathrm{i}(\alpha_1 k+\alpha_2 k^2+\alpha_3 k^3)s)\right. \\
&\quad \left.\times |\mathcal{L}(s)\{v\}|^2\mathcal{L}(s)\{v\}\,\mathrm{d}s\right)\mathrm{d}t \\
&= \int_{\mathbb{R}} K(s)\left(\int_{\mathbb{R}} \exp(-\mathrm{i}kt)\,\exp(-\mathrm{i}(\alpha_1 k+\alpha_2 k^2+\alpha_3 k^3)s)\right. \\
&\quad \left.\times |\mathcal{L}(s)\{v\}|^2\mathcal{L}(s)\{v\}\,\mathrm{d}t\right)\mathrm{d}s \\
&= \int_{\mathbb{R}} K(s)\Big(\mathcal{L}(-s)\{|\mathcal{L}(s)\{v\}|^2\mathcal{L}(s)\{v\}\}\Big)^{\wedge}\Big)\,\mathrm{d}s
\end{aligned}$$

and it is now clear that

$$\mathscr{N}\{v\} = \int_{\mathbb{R}} K(s)\mathscr{L}(-s)\{|\mathscr{L}(s)\{v\}|^2\mathscr{L}(s)\{v\}\}\,\mathrm{d}s.$$

We note here that $\mathscr{L}$ is an isometry on $H^s(\mathbb{R})$ for all $s \in \mathbb{R}$, as this property will become important in later estimates.

3. Hamiltonian structure

We claim that the functional

$$\mathscr{H}\{v\} = \int_{\mathbb{R}} |v_t|^2\,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} K(s)|\mathscr{L}(s)\{v\}|^4\,\mathrm{d}t\,\mathrm{d}s \tag{4}$$

is C^1 on $H^1(\mathbb{R})$, and moreover, that it is a Hamiltonian for Eq. (1), as

$$v_z = J\nabla\mathscr{H},$$

where J is the skew-symmetric operator $-i$. First, we note that $\mathscr{H}$ is well defined on $H^1(\mathbb{R})$. We apply the Sobolev-type inequality [6]

$$\|u\|^4_{L^4(\mathbb{R})} \leqslant C\|u_t\|_{L^2(\mathbb{R})}\|u\|^3_{L^2(\mathbb{R})} \tag{5}$$

and the facts that $K(s) \in L^1$ and $\mathscr{L}$ is an isometry on H^1 to show that

$$\begin{aligned}
|\mathscr{H}\{v\}| &\leqslant \int_{\mathbb{R}} |v_t|^2\,\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |K(s)||\mathscr{L}(s)\{v\}|^4\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_{\mathbb{R}} |v_t|^2\,\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}} |K(s)|\left(\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^4\,\mathrm{d}t\right)\mathrm{d}s \\
&\leqslant \int_{\mathbb{R}} |v_t|^2\,\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}} |K(s)|(C\|v(t)\|^3_{L^2(\mathbb{R})}\|v_t(t)\|_{L^2(\mathbb{R})})\,\mathrm{d}s \\
&= \int_{\mathbb{R}} |v_t|^2\,\mathrm{d}t + \left(\frac{C}{2}\|v(t)\|^3_{L^2(\mathbb{R})}\|v_t(t)\|_{L^2(\mathbb{R})}\right)\int_{\mathbb{R}} |K(s)|\,\mathrm{d}s \leqslant C.
\end{aligned}$$

We now calculate $\nabla\mathscr{H}$ directly. Let $v, u \in H^1(\mathbb{R})$ and consider

$$
\begin{aligned}
&\langle \nabla\mathscr{H}(v), u\rangle \\
&\quad = \lim_{\varepsilon\to 0} \frac{\mathscr{H}(v+\varepsilon u) - \mathscr{H}(v)}{\varepsilon} \\
&\quad = \lim_{\varepsilon\to 0} \frac{\int_{\mathbb{R}} |(v+\varepsilon u)_t|^2 \mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} K(s)|\mathscr{L}(s)\{(v+\varepsilon u)\}|^4 \mathrm{d}t\mathrm{d}s - \int_{\mathbb{R}} |v_t|^2 \mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} K(s)|\mathscr{L}(s)\{v\}|^4 \mathrm{d}t\mathrm{d}s}{\varepsilon} \\
&\quad = \int_{\mathbb{R}} v_t \bar{u}_t \,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} K(s)\left(\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^2 \mathscr{L}(s)\{v\}\overline{\mathscr{L}(s)\{u\}}\,\mathrm{d}t\right)\mathrm{d}s + c.c. \\
&\quad = -\int_{\mathbb{R}} v_{tt}\bar{u}\,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} K(s)\left(\int_{\mathbb{R}} \mathscr{L}(-s)\{|\mathscr{L}(s)\{v\}|^2 \mathscr{L}(s)\{v\}\}\bar{u}\,\mathrm{d}t\right)\mathrm{d}s + c.c. \\
&\quad = -\int_{\mathbb{R}} v_{tt}\bar{u}\,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(s)\mathscr{L}(-s)\{|\mathscr{L}(s)\{v\}|^2 \mathscr{L}(s)\{v\}\}\,\mathrm{d}s\right)\bar{u}\,\mathrm{d}t + c.c. \\
&\quad = -\int_{\mathbb{R}} \left(v_{tt} + \frac{1}{2}\int_{\mathbb{R}} K(s)\mathscr{L}(-s)\{|\mathscr{L}(s)\{v\}|^2 \mathscr{L}(s)\{v\}\}\,\mathrm{d}s\right)\bar{u}\,\mathrm{d}t + c.c.
\end{aligned}
$$

which gives the weak form of the Euler–Lagrange equation (1). Now

$$
\begin{aligned}
|\langle \nabla\mathscr{H}(v), u\rangle| &\leqslant \int_{\mathbb{R}} |v_t u_t|\,\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |K(s)||\mathscr{L}(s)\{v\}|^2|\mathscr{L}(s)\{v\}||\mathscr{L}\{u\}|\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_{\mathbb{R}} |v_t u_t|\,\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}} |K(s)|\left(\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^3|\mathscr{L}\{u\}|\,\mathrm{d}t\right)\mathrm{d}s \\
&\leqslant \|v\|_{H^1}\|u\|_{H^1} + \frac{1}{2}\int_{\mathbb{R}} |K(s)|\|\mathscr{L}(s)\{u\}\|_{L^\infty}\|\mathscr{L}(s)\{v\}\|_{L^\infty} \\
&\qquad \times \int_{\mathbb{R}} (|\mathscr{L}(s)\{v\}|^2\,\mathrm{d}t)\,\mathrm{d}s \\
&\leqslant \|v\|_{H^1}\|u\|_{H^1} + \frac{1}{2}\int_{\mathbb{R}} |K(s)|\|\mathscr{L}(s)\{u\}\|_{H^1}\|\mathscr{L}(s)\{v\}\|_{H^1} \\
&\qquad \times \left(\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^2\,\mathrm{d}t\right)\mathrm{d}s \\
&= \|v\|_{H^1}\|u\|_{H^1} + \frac{1}{2}\|u\|_{H^1}\|v\|_{H^1}\int_{\mathbb{R}} |K(s)|\int_{\mathbb{R}} |v|^2\,\mathrm{d}t\,\mathrm{d}s \\
&\leqslant C\|v\|_{H^1}\|u\|_{H^1}.
\end{aligned}
$$

This shows that $\nabla\mathscr{H}$ is bounded on H^1.

Using similar estimates, it can be shown that

$$
|\mathscr{H}(v+u) - \mathscr{H}(v) - \langle \nabla\mathscr{H}(v), u\rangle| = o(\|u\|_{H^1})
$$

as $\|u\|_{H^1} \to 0$, so that $\mathscr{H}$ is differentiable, and also that for $u, v, w \in H^1$

$$|\langle(\nabla\mathscr{H}(v) - \nabla\mathscr{H}(u)), w\rangle| \leqslant C\|v-u\|_{H^1}\|w\|_{H^1},$$

so that

$$\|(\nabla\mathscr{H}(v) - \nabla\mathscr{H}(u))\|_{H^{-1}} = \sup_{w\neq 0} \frac{|\langle(\nabla\mathscr{H}(v) - \nabla\mathscr{H}(u)), w\rangle|}{\|w\|_{H^1}} \to 0$$

as $\|v-u\|_{H^1} \to 0$, i.e. $\mathscr{H}$ is C^1 on H^1.

4. Properties of $\mathscr{H}$

Here we present properties of $\mathscr{H}(v)$ that are essential to the minimization argument.

1. *$\mathscr{H}(v)$ is bounded from below in $\mathscr{A}_\lambda$*

Proof. An application of the Sobolev-type inequality gives

$$\mathscr{H}(v) \geqslant \|v_t\|_{L^2}^2 - M\lambda^{3/2}\|v_t\|_{L^2} = (\|v_t\|_{L^2} - c_1)^2 - c_2 > -\infty \tag{6}$$

for all $v \in H^1(\mathbb{R})$, so $\mathscr{I}_\lambda = \inf_{\mathscr{A}_\lambda}\mathscr{H}$ exists with $\mathscr{I}_\lambda > -\infty$. $\square$

2. $\mathscr{I}_\lambda = \inf_{v\in\mathscr{A}_\lambda}\mathscr{H}(v) < 0$

Proof. Let $v \in \mathscr{A}_\lambda$ and consider the rescaled function $v_\gamma(t) = \gamma^{1/2}v(\gamma t) \in \mathscr{A}_\lambda$. Evaluating $\mathscr{H}$ at v_γ gives

$$\begin{aligned}\mathscr{H}(v_\gamma) &= \int_{\mathbb{R}}\left|\frac{\partial v_\gamma}{\partial t}\right|^2 \mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |\mathscr{L}(s)\{v_\gamma\}|^4\,\mathrm{d}t\,\mathrm{d}s \\ &= \gamma^2\int_{\mathbb{R}}\left|\frac{\partial v}{\partial t'}\right|^2 \mathrm{d}t' - \frac{1}{2}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |\mathscr{L}(s)\{v_\gamma\}|^4\,\mathrm{d}t\,\mathrm{d}s,\end{aligned}$$

where $t' = \gamma t$. Using a scaling property of the Fourier transform, we have

$$\begin{aligned}\mathscr{L}(s)\{v_\gamma\} &= \int_{\mathbb{R}} \exp(\mathrm{i}kt)\exp(\mathrm{i}(\alpha_1 k + \alpha_2 k^2 + \alpha_3 k^3)s)\left(\gamma^{-1/2}\hat{v}\left(\frac{k}{\gamma}\right)\right)\mathrm{d}k \\ &= \gamma^{1/2}\int_{\mathbb{R}} \exp(\mathrm{i}k'(\gamma t))\exp(\mathrm{i}(\alpha_1(\gamma k') + \alpha_2(\gamma k')^2 + \alpha_3(\gamma k')^3)s)\hat{v}(k')\,\mathrm{d}k' \\ &= \gamma^{1/2}u_\gamma(s, t'),\end{aligned}$$

where $k' = \frac{k}{\gamma}$ and $u_\gamma(s, t')$ solves the linear equation

$$i\,\frac{\partial u_\gamma}{\partial s} + i\alpha_1\gamma\,\frac{\partial u_\gamma}{\partial t'} - \alpha_2\gamma^2\,\frac{\partial^2 u_\gamma}{\partial t'^2} - i\alpha_3\gamma^3\,\frac{\partial^3 u_\gamma}{\partial t'^3} = 0$$

with initial data $v(t')$. We may write

$$u_\gamma(s, t') = \tilde{\mathscr{L}}_\gamma(s)\{v\},$$

where $\tilde{\mathscr{L}}_\gamma$ is an isometry on $H^s(\mathbb{R})$ and thus

$$\begin{aligned}\mathscr{H}(v_\gamma) &= \gamma^2 \int_{\mathbb{R}} \left|\frac{\partial v}{\partial t'}\right|^2 \mathrm{d}t' - \frac{\gamma^2}{2}\int_{\mathbb{R}} K(s) \int_{\mathbb{R}} |\tilde{\mathscr{L}}_\gamma(s)\{v\}|^4\,\mathrm{d}t\,\mathrm{d}s \\ &= \gamma^2 \int_{\mathbb{R}} \left|\frac{\partial v}{\partial t'}\right|^2 \mathrm{d}t' - \frac{\gamma}{2}\int_{\mathbb{R}} K(s) \int_{\mathbb{R}} |\tilde{\mathscr{L}}_\gamma(s)\{v\}|^4\,\mathrm{d}t'\,\mathrm{d}s.\end{aligned}$$

Now the integral of $K(s)\int_{\mathbb{R}} |\tilde{\mathscr{L}}_\gamma(s)\{v\}|^4\,\mathrm{d}t'$ over any finite interval in s approaches a constant as $\gamma \to 0$, since for any s finite, $\tilde{\mathscr{L}}_\gamma(s)\{v\} \to v(t')$ in H^1 as $\gamma \to 0$, which, by (5), implies convergence in L^4. Also, $\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |\tilde{\mathscr{L}}_\gamma(s)\{v\}|^4\,\mathrm{d}t'\,\mathrm{d}s$ is finite for any γ, which follows from previous estimates. Thus for every $\varepsilon > 0$, there exists some $N(\varepsilon) > 0$ independent of γ such that

$$\int_{\mathbb{R}\setminus[-N(\varepsilon),N(\varepsilon)]} K(s) \int_{\mathbb{R}} |\tilde{\mathscr{L}}_\gamma(s)\{v\}|^4\,\mathrm{d}t'\,\mathrm{d}s < \varepsilon.$$

If we write

$$\begin{aligned}\mathscr{H}(v_\gamma) = \gamma^2 \int_{\mathbb{R}} \left|\frac{\partial v}{\partial t'}\right|^2 \mathrm{d}t' - \frac{\gamma}{2}\Bigg(&\int_{[-N(\varepsilon),N(\varepsilon)]} K(s) \int_{\mathbb{R}} |\tilde{\mathscr{L}}_\gamma(s)\{v\}|^4\,\mathrm{d}t' \\ &+ \int_{\mathbb{R}\setminus[-N(\varepsilon),N(\varepsilon)]} K(s) \int_{\mathbb{R}} |\tilde{\mathscr{L}}_\gamma(s)\{v\}|^4\,\mathrm{d}t'\,\mathrm{d}s\Bigg)\end{aligned}$$

and take $\gamma \to 0$ at a rate say $\sim \frac{1}{N(\varepsilon)^2}$ as $\varepsilon \to 0$, then to leading order we get

$$\mathscr{H}(v_\gamma) = C_1\gamma^2 - C_2\gamma$$

which is negative for γ small enough. □

3. *$\mathscr{H}$ is subadditive*:
   If $\mathscr{I}_\lambda = \inf_{v\in\mathscr{A}_\lambda} \mathscr{H}(v)$ then $\mathscr{I}_{\lambda_1+\lambda_2} < \mathscr{I}_{\lambda_1} + \mathscr{I}_{\lambda_2}$

Claim. *For $\theta > 1$, $\mathscr{I}_{\theta\lambda} < \theta\mathscr{I}_\lambda$*

Proof of claim. Now

$$\begin{aligned}\mathscr{I}_{\theta\lambda} &= \inf_{v\in\mathscr{A}_{\theta\lambda}} \mathscr{H}(v)\\ &= \inf_{w\in\mathscr{A}_{\lambda}} \mathscr{H}(\sqrt{\theta}w)\end{aligned}$$

since

$$\|w\|_{L^2}^2 = \lambda \Rightarrow \|\sqrt{\theta}w\|_{L^2}^2 = \theta\lambda.$$

The claim follows from

$$\begin{aligned}\mathscr{I}_{\theta\lambda} \leqslant \mathscr{H}(\sqrt{\theta}w) &= \int_{\mathbb{R}} |(\sqrt{\theta}w)_t|^2\,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |\mathscr{L}\{\sqrt{\theta}w\}|^4\,\mathrm{d}t\,\mathrm{d}s\\ &= \theta\int_{\mathbb{R}} |w_t|^2\,\mathrm{d}t - \frac{\theta^2}{2}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |\mathscr{L}\{w\}|^4\,\mathrm{d}t\,\mathrm{d}s\\ &< \theta\left(\int_{\mathbb{R}} |w_t|^2\,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |\mathscr{L}\{w\}|^4\,\mathrm{d}t\,\mathrm{d}s\right)\end{aligned}$$

for $\theta > 1$ and $w \in \mathscr{A}_{\lambda}$. □

Proof of subadditivity. If we set $\lambda_1 = \alpha\lambda_2$ with $\alpha < 1$, we have

$$\mathscr{I}_{\lambda_1+\lambda_2} = \mathscr{I}_{\alpha\lambda_2+\lambda_2} < (\alpha+1)\mathscr{I}_{\lambda_2} = \alpha\mathscr{I}_{(\alpha^{-1}\lambda_1)} + \mathscr{I}_{\lambda_2} < \mathscr{I}_{\lambda_1} + \mathscr{I}_{\lambda_2}. \qquad \square$$

4. *A minimizing sequence for $\mathscr{H}(v)$ cannot vanish or split in the sense of Lions' concentration compactness lemma* [15].

Proof. Let v_k be a minimizing sequence for $\mathscr{H}(v)$ and first assume that a subsequence v_{m_k} vanishes in the sense of the concentration-compactness lemma, i.e.

For any $R > 0$

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}}\int_{y-R}^{y+R} |v_{m_k}|^2\,\mathrm{d}t \to 0.$$

Since v_k is a minimizing sequence, for some $k_0 \geqslant 1$ we have that

$$\int_{\mathbb{R}} |(v_k)_t|^2\,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |\mathscr{L}\{v_k\}|^4\,\mathrm{d}t\,\mathrm{d}s < 0,$$

for $k \geqslant k_0$, and so for k large enough

$$\int_{\mathbb{R}}\left(K(s)\int_{\mathbb{R}} |\mathscr{L}\{v_k\}|^4\,\mathrm{d}t\right)\mathrm{d}s > 0.$$

Thus on some set of positive measure

$$K(s)\int_{\mathbb{R}} |\mathscr{L}(s)\{v_k\}|^4\,\mathrm{d}t > 0$$

so for some finite s^*, we have

$$K(s^*)\int_{\mathbb{R}} |\mathscr{L}(s^*)\{v_k\}|^4\,\mathrm{d}t > 0.$$

Since $K(s)$ is positive, this gives

$$\int_{\mathbb{R}} |\mathscr{L}(s^*)\{v_k\}|^4\,\mathrm{d}t > 0. \tag{7}$$

Now a lemma of Cazenave [7] gives that

$$\int_{\mathbb{R}} |u|^4\,\mathrm{d}t \leqslant C\|u\|_{H^1}^2 \sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |u|^2\,\mathrm{d}t$$

for any $u \in H^1(\mathbb{R})$. Since $\mathscr{L}(s^*)\{v_k\} \in H^1(\mathbb{R})$, this lemma, in combination with (7), gives

$$\begin{aligned} 0 < \int_{\mathbb{R}} |\mathscr{L}(s^*)\{v_k\}|^4\,\mathrm{d}t &\leqslant C\|\mathscr{L}(s^*)\{v_k\}\|_{H^1}^2 \sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |\mathscr{L}(s^*)\{v_k\}|^2\,\mathrm{d}t \\ &= C\|v_k\|_{H^1}^2 \sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |\mathscr{L}(s^*)\{v_k\}|^2\,\mathrm{d}t \end{aligned}$$

so

$$\sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |\mathscr{L}(s^*)\{v_k\}|^2\,\mathrm{d}t > 0. \tag{8}$$

Now $\mathscr{L}$ is the solution operator for the linear equation (3), so we may apply the following localization lemma to show that the initial data cannot vanish.

Lemma 4.1. *Consider the linear equation*

$$iu_s + i\alpha_1 u_t - \alpha_2 u_{tt} - i\alpha_3 u_{ttt} = 0 \tag{9}$$

with $u \in H^1(\mathbb{R})$, $\|u\|_{L^2(\mathbb{R})} = 1$, *and* $\alpha_j \in \mathbb{R}$. *Let* $u_n(s,t)$ *be a sequence of solutions of* (9) *and define*

$$\varepsilon_n(s) = \sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |u_n(s,t)|^2\,\mathrm{d}t.$$

If $u_n(0,t)$ *is vanishing initial data* ($\lim_{n\to\infty} \varepsilon_n(0) = 0$) *with the constraint* $\|u_n\|_{L^2(\mathbb{R})} = 1$, *then the sequence of the solutions* $u_n(s,t)$ *is also vanishing* ($\lim_{n\to\infty} \varepsilon_n(s) = 0$).

Proof. Let $\chi_m(t)$ be a smooth approximation to the characteristic function on the interval $[-m, m]$, with the property that $|\partial_t \chi_m| < \frac{C}{m}$, $\chi_m(t) = 1$ if $|t| \leqslant 1$, and $\chi(t) = 0$ if $|t| \geqslant m$. Multiplying (9) by $\bar{u}\chi_m(t)$, its conjugate by $u\chi_m(t)$, subtracting and integrating over t yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}} \chi_m |u|^2 \,\mathrm{d}t = -2\alpha_1 \operatorname{Re}\int_{\mathbb{R}} u_t \bar{u}\chi_m \,\mathrm{d}t + 2\alpha_2 \operatorname{Im}\int_{\mathbb{R}} \chi_m \bar{u} u_{tt} \,\mathrm{d}t + 2\alpha_3 \operatorname{Re}\int_{\mathbb{R}} \chi_m \bar{u} u_{ttt} \,\mathrm{d}t.$$

Now

$$\operatorname{Re}\int_{\mathbb{R}} u_t \bar{u}\chi_m \,\mathrm{d}t = -\frac{1}{2}\int_{\mathbb{R}} \chi_m' |u|^2 \,\mathrm{d}t,$$

$$\operatorname{Im}\int_{\mathbb{R}} \chi_m \bar{u} u_{tt} \,\mathrm{d}t = -\operatorname{Im}\int_{\mathbb{R}} (\chi_m \bar{u}_t + \chi_m' \bar{u}) u_t \,\mathrm{d}t = -\operatorname{Im}\int_{\mathbb{R}} \chi_m' \bar{u} u_t \,\mathrm{d}t$$

and

$$\begin{aligned}
\operatorname{Re}\int_{\mathbb{R}} \chi_m \bar{u} u_{ttt} \,\mathrm{d}t &= -\operatorname{Re}\int_{\mathbb{R}} (\chi_m \bar{u}_t + \chi_m' \bar{u}) u_{tt} \,\mathrm{d}t \\
&= -\operatorname{Re}\int_{\mathbb{R}} \chi_m \bar{u}_t u_{tt} \,\mathrm{d}t - \operatorname{Re}\int_{\mathbb{R}} \chi_m' \bar{u} u_{tt} \,\mathrm{d}t \\
&= \frac{1}{2}\int_{\mathbb{R}} \frac{\mathrm{d}|u_t|^2}{\mathrm{d}t}\chi_m + \operatorname{Re}\int_{\mathbb{R}} (\chi_m'' \bar{u} + \chi_m' \bar{u}_t) u_t \,\mathrm{d}t \\
&= \frac{3}{2}\int_{\mathbb{R}} |u_t|^2 \chi_m' \,\mathrm{d}t + \operatorname{Re}\int_{\mathbb{R}} \chi_m'' \bar{u} u_t \,\mathrm{d}t.
\end{aligned}$$

Overall,

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}} \chi_m |u|^2 \,\mathrm{d}t = \alpha_1 \int_{\mathbb{R}} \chi_m' |u|^2 \,\mathrm{d}t - 2\alpha_2 \operatorname{Im}\int_{\mathbb{R}} \chi_m' \bar{u} u_t \,\mathrm{d}t + 2\alpha_3 \left(\frac{3}{2}\int_{\mathbb{R}} |u_t|^2 \chi_m' \,\mathrm{d}t + \operatorname{Re}\int_{\mathbb{R}} \chi_m'' \bar{u} u_t \,\mathrm{d}t\right)$$

and integrating from 0 to s gives

$$\begin{aligned}
\int_{\mathbb{R}} \chi_m |u(s,t)|^2 \,\mathrm{d}t &= \int_{\mathbb{R}} \chi_m |u(0,t)|^2 \,\mathrm{d}t \\
&\quad + \int_0^s \left(\alpha_1 \int_{\mathbb{R}} \chi_m' |u|^2 \,\mathrm{d}t - 2\alpha_2 \operatorname{Im}\int_{\mathbb{R}} \chi_m' \bar{u} u_t \,\mathrm{d}t \right. \\
&\quad \left. + 2\alpha_3 \left(\frac{3}{2}\int_{\mathbb{R}} |u_t|^2 \chi_m' \,\mathrm{d}t + \operatorname{Re}\int_{\mathbb{R}} \chi_m'' \bar{u} u_t \,\mathrm{d}t\right)\right) \mathrm{d}z' \\
&\leqslant \int_{\mathbb{R}} \chi_m |u(0,t)|^2 \,\mathrm{d}t + C_m(\|u\|_{H^1}, \|\chi_m'\|_{L^\infty}, \|\chi_m''\|_{L^\infty}, \alpha_j), \quad (10)
\end{aligned}$$

where $C_m \to 0$ as $m \to \infty$.

Let $u_n(s,t)$ denote a sequence of solutions of (9) with vanishing initial data, i.e.

$$\varepsilon_n(0) = \sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |u_n(0,t)|^2 \,\mathrm{d}t \to 0.$$

If $\varepsilon_n(z) \leqslant \varepsilon_n(0)$ then we are done, so let $\varepsilon_n(z) > \varepsilon_n(0)$. Choosing $\chi_{m_n}(* - t_n)$ such that it is centered with respect to $u_n(s,t)$, we have

$$\int_{\mathbb{R}} \chi_{m_n} |u_n(s,t)|^2 \,\mathrm{d}t \geqslant \varepsilon_n(s)$$

and also

$$\int_{\mathbb{R}} \chi_{m_n} |u_n(0,t)|^2 \,\mathrm{d}t \leqslant 2m_n\varepsilon_n(0)$$

and taking the limit $m_n \to \infty$ with $m_n \sim \sqrt{\frac{1}{\varepsilon_n(0)}}$ in (10) gives the result. $\square$

Combining this lemma with (8), we have

$$\sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |v_k|^2 \,\mathrm{d}t \geqslant \sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |v_k|^2 \,\mathrm{d}t \geqslant \sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |\mathscr{L}(s^*)\{v_k\}|^2 \,\mathrm{d}t > 0$$

contradicting the assumption that a subsequence v_{m_k} vanishes.

Now assume that v_{m_k} splits: There exists $0 < \gamma < \lambda$ such that for any $\varepsilon > 0$ there exists k_0 and two sequences u_k, w_k with compact support so that for $k \geqslant k_0$

$$\|u_k\|_{H^1} + \|w_k\|_{H^1} \leqslant 4 \sup_{k\in\mathbf{N}} \|v_{m_k}\|_{H^1},$$

$$\|v_{m_k} - (u_k + w_k)\|_{L^2} \leqslant 2\varepsilon,$$

$$|\|u_k\|_{L^2} - \gamma| \leqslant \varepsilon |\|w_k\|_{L^2} - (\lambda - \gamma)| \leqslant \varepsilon,$$

$$\left\|\frac{\partial u_k}{\partial t}\right\|_{L^2} + \left\|\frac{\partial w_k}{\partial t}\right\|_{L^2} \leqslant \left\|\frac{\partial v_{m_k}}{\partial t}\right\|_{L^2} + \varepsilon$$

and $\operatorname{dist}(\operatorname{supp}(v_k), \operatorname{supp}(w_k)) > 2\varepsilon^{-1}$. To rule out splitting, it is enough to show that

$$\mathscr{H}(v_{m_k}) > \mathscr{H}(w_k) + \mathscr{H}(u_k) + \alpha(\varepsilon),$$

where $\alpha(\varepsilon)$ is independent of k and goes to 0 as $\varepsilon \to 0$, which causes $\mathscr{H}(v_{m_k})$ to violate subadditivity.

We directly evaluate $\mathscr{H}\{v_{m_k}\}$:

$$\mathscr{H}(v_{m_k}) = \int_{\mathbb{R}} |(u_k + w_k + h_k)_t|^2 \,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} K(s) \int_{\mathbb{R}} |\mathscr{L}\{u_k + w_k + h_k\}|^4 \,\mathrm{d}t\,\mathrm{d}s,$$

where

$$\|h_k\|_{L^2}^2 < \varepsilon$$

and we have suppressed the notation $\mathscr{L}(s)$.

Expanding the terms, this can be rewritten

$$\begin{aligned}
\mathscr{H}(v_{m_k}) = {} & \mathscr{H}(u_k) + \mathscr{H}(w_k) \\
& + 2\,\mathrm{Re}\int_{\mathbb{R}} (\partial_t \overline{u}_k \partial_t w_k + \partial_t \overline{u}_k \partial_t h_k + \partial_t \overline{w}_k \partial_t h_k + |\partial_t h_k|^2)\,\mathrm{d}t \\
& - \mathrm{Re}\int_{\mathbb{R}}\int_{\mathbb{R}} K(s)\Big(2|\mathscr{L}\{u_k + w_k\}|^2|\mathscr{L}\{h_k\}|^2 + \frac{1}{2}|\mathscr{L}\{h_k\}|^4 \\
& + 2|\mathscr{L}\{u_k + w_k\}|^2(\mathscr{L}\{u_k + w_k\})(\mathscr{L}\{h_k\}) + (\mathscr{L}\{u_k + w_k\})^2(\overline{\mathscr{L}\{h_k\}})^2 \\
& + 2\mathscr{L}\{u_k + w_k\}|\mathscr{L}\{h_k\}|^2\overline{\mathscr{L}h_k}\,\mathrm{d}t\Big)\,\mathrm{d}s \\
& + \int_{\mathbb{R}} K(s)\int_{\mathbb{R}} (2|\mathscr{L}\{u_k\}|^2|\mathscr{L}\{w_k\}|^2 + 2|\mathscr{L}\{u_k\}|^2\mathscr{L}\{u_k\}\mathscr{L}\{\overline{w}_k\} \\
& + (\mathscr{L}\{u_k\})^2(\mathscr{L}\{\overline{w}_k\})^2 + 2|\mathscr{L}\{w_k\}|^2\mathscr{L}\{u_k\}\mathscr{L}\{\overline{w}_k\})\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

We proceed as in [25]. The terms

$$2\,\mathrm{Re}\int_{\mathbb{R}} (\partial_t \overline{u}_k \partial_t w_k + \partial_t \overline{u}_k \partial_t h_k + \partial_t \overline{w}_k \partial_t h_k + |\partial_t h_k|^2)\,\mathrm{d}t$$

can be estimated from below by $-C_1\varepsilon$ with Hölder's inequality.

The terms

$$\begin{aligned}
& -\mathrm{Re}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} (2|\mathscr{L}\{u_k + w_k\}|^2|\mathscr{L}\{h_k\}|^2 + \frac{1}{2}|\mathscr{L}\{h_k\}|^4 \\
& \quad + 2|\mathscr{L}\{u_k + w_k\}|^2(\mathscr{L}\{u_k + w_k\})(\mathscr{L}\{h_k\}) + (\mathscr{L}\{u_k + w_k\})^2(\overline{\mathscr{L}\{h_k\}})^2 \\
& \quad + 2\mathscr{L}\{u_k + w_k\}|\mathscr{L}\{h_k\}|^2\overline{\mathscr{L}h_k})\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}$$

are all estimated from below by Hölder's inequality and the Sobolev inequality, yielding a lower bound of the form $-C_2(\varepsilon)$. The remaining terms

$$\begin{aligned}
& \int_{\mathbb{R}} K(s)\int_{\mathbb{R}} (2|\mathscr{L}\{u_k\}|^2|\mathscr{L}\{w_k\}|^2 + 2|\mathscr{L}\{u_k\}|^2\mathscr{L}\{u_k\}\mathscr{L}\{\overline{w}_k\} \\
& \quad + (\mathscr{L}\{u_k\})^2(\mathscr{L}\{\overline{w}_k\})^2 + 2|\mathscr{L}\{w_k\}|^2\mathscr{L}\{u_k\}\mathscr{L}\{\overline{w}_k\})\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}$$

are also estimated from below similarly, but require the application of the following lemma, which is a straightforward consequence of the localization lemma:

Lemma. *Let* $t_c = \frac{t_1+t_2}{2}$ *and* $A_j = \{|t - t_j| \leqslant t_c\}$, $j = 1, 2$. *Then*

$$\int_{A_j} |\mathscr{L}\{w_k\}|^2 \, \mathrm{d}\tau \leqslant C\varepsilon,$$

$$\int_{\mathbb{R}\setminus A_j} |\mathscr{L}\{u_k\}|^2 \, \mathrm{d}\tau \leqslant C\varepsilon.$$

After application of the lemma, we have

$$\mathscr{H}(v_{m_k}) > \mathscr{H}(w_k) + \mathscr{H}(u_k) + \alpha(\varepsilon),$$

where $\alpha(\varepsilon)$ is independent of k and goes to 0 as $\varepsilon \to 0$. Thus splitting causes $\mathscr{H}$ to violate subadditivity, a contradiction.

5. Minimization

We now prove the first part of Theorem 1.1, demonstrating that the Hamiltonian (4) possesses a minimizer in $\mathscr{A}_\lambda$.

Proof of Theorem 1.1 (*existence*). Now in the previous section, it was shown that $-\infty < \mathscr{I}_\lambda < 0$, which will help rule out loss of compactness of minimizing sequences. Let $\{v_k\}$ be a minimizing sequence for $\mathscr{H}(v)$. Now from (6), we can deduce that $\|v_k\|_{H^1}$ must be bounded, so by Alaoglu's theorem, there exists a weakly convergent subsequence in $H^1(\mathbb{R})$, v_{k_m}. We will prove strong convergence of v_{k_m} to a minimizer in $H^1(\mathbb{R})$, and first establish strong convergence in $L^2(\mathbb{R})$.

We use the fact that a minimizing sequence for $\mathscr{H}(v)$ cannot vanish or split in the sense of Lions' concentration-compactness lemma and conclude that the minimizing sequence remains localized as $m \to \infty$, i.e. for any $\varepsilon > 0$ there exists an $R > 0$ such that

$$\int_{-R}^{+R} |w_m(t)|^2 \, \mathrm{d}t > \lambda - \varepsilon, \tag{11}$$

where $w_m(t) = v_{k_m}(t - t_m)$.

Now $w_m \rightharpoonup w^*$ for some $w^* \in H^1(\mathbb{R})$. For any $R > 0$, the embedding $H^1(\mathbb{R}) \hookrightarrow L^2([-R, R])$ is compact and we have

$$\lim_{m\to\infty} \int_{-R}^{+R} |w_m|^2 \, \mathrm{d}t = \int_{-R}^{R} |w^*|^2 \, \mathrm{d}t.$$

Together with (11), this implies

$$\int_{\mathbb{R}} |w^*|^2 \, \mathrm{d}t > \lambda - \varepsilon \quad \text{for any } \varepsilon > 0$$

and therefore

$$\int_{\mathbb{R}} |w^*|^2 \, \mathrm{d}t = \lambda.$$

This norm convergence, along with weak convergence in $L^2(\mathbb{R})$, gives strong convergence of w_m to w^* in $L^2(\mathbb{R})$.

We now show that w^* is a minimizer, and a posteriori that $\partial_t w_m \to \partial_t w^*$ in $L^2(\mathbb{R})$. Since w_m converges weakly to w^* in H^1 and the Sobolev norm $\| * \|_{H^1}$ is weakly lower semi-continuous, we have that

$$\|w^*\|_{H^1} \leqslant \liminf_{m\to\infty} \|w_m\|_{H^1}.$$

This, together with $w_m \to w^*$ in $L^2(\mathbb{R})$, implies that

$$\|\partial_t w^*\|_{L^2} \leqslant \liminf_{m\to\infty} \|\partial_t w_m\|_{L^2}. \tag{12}$$

Now we show that

$$\int_{\mathbb{R}} K(s) \int_{\mathbb{R}} |\mathscr{L}(s)\{w_m\}|^4 \, \mathrm{d}t \, \mathrm{d}s \to \int_{\mathbb{R}} K(s) \int_{\mathbb{R}} |\mathscr{L}(s)\{w^*\}|^4 \, \mathrm{d}t \, \mathrm{d}s.$$

For any u_m and $u^* \in H^1(\mathbb{R})$, the Sobolev-type inequality, linearity of $\mathscr{L}$, and the fact that $\mathscr{L}$ is an isometry on $H^1(\mathbb{R})$ gives,

$$\int_{\mathbb{R}} |\mathscr{L}(s)\{u_m\} - \mathscr{L}(s)\{u^*\}|^4 \, \mathrm{d}t \leqslant C \left(\int_{\mathbb{R}} |u_m - u^*|^2 \, \mathrm{d}t \right)^{3/2},$$

where C depends on $\|\frac{\partial(u_m - u^*)}{\partial t}\|_{L^2(\mathbb{R})}$.

Thus for u_m and $u^* \in H^1(\mathbb{R})$, C is bounded and if $u_m \to u^*$ in $L^2(\mathbb{R})$, then $\mathscr{L}\{u_m\} \to \mathscr{L}\{u^*\}$ in $L^4(\mathbb{R})$ with

$$\|\mathscr{L}(s)\{w^*\}\|_{L^4} = \lim_{m\to\infty} \|\mathscr{L}(s)\{w_m\}\|_{L^4}.$$

Thus

$$\begin{aligned} &\int_{\mathbb{R}} K(s) \int_{\mathbb{R}} |\mathscr{L}(s)\{w_m\}|^4 \, \mathrm{d}t \, \mathrm{d}s - \int_{\mathbb{R}} K(s) \int_{\mathbb{R}} |\mathscr{L}(s)\{w^*\}|^4 \, \mathrm{d}t \, \mathrm{d}s \\ &\quad = \int_{\mathbb{R}} K(s) \left(\int_{\mathbb{R}} |\mathscr{L}(s)\{w_m\}|^4 - |\mathscr{L}(s)\{w^*\}|^4 \, \mathrm{d}t \right) \mathrm{d}s \to 0. \end{aligned}$$

Combining the above result with (12), we have $\mathscr{H}(w^*) \leqslant \liminf_{m\to\infty} \mathscr{H}(w_m)$. This can only happen if $\mathscr{H}(w^*) = \lim_{m\to\infty} \mathscr{H}(w_m)$, i.e. $w^* \in \mathscr{A}_\lambda$ is the minimizer of $\mathscr{H}$ over $\mathscr{A}_\lambda$. Moreover, $\|\partial_t w^*\|_{L^2} = \lim_{m\to\infty} \|\partial_t w_m\|_{L^2}$, and again, this norm convergence, with weak convergence, implies strong convergence of $\partial_t w_m$ to $\partial_t w^*$ in $L^2(\mathbb{R})$, so $w_m \to w^*$ in $H^1(\mathbb{R})$. $\square$

6. Stability

It is clear that the minimizer is not unique, as any translation $v(\cdot+\tau_0)$, $\tau_0 \in \mathbb{R}$, or rotation $e^{i\theta}v$, $\theta \in \mathbb{R}$, of the minimizer is also a solution of the constrained minimization problem. If we consider the class of ground state solutions $\mathscr{S}_\lambda = \{v_g \in \mathscr{A}_\lambda, \mathscr{H}(v_g) = I_\lambda\}$, then using the strong convergence of minimizing sequences and conservation laws for (1), one can show that the minimizer is stable in the following orbital sense. The following is a standard argument [7].

Theorem 6.1. *Let $\mathscr{S}_\lambda$ be the set of ground states $\mathscr{S}_\lambda = \{v_g \in \mathscr{A}_\lambda, \mathscr{H}(v_g) = \mathscr{I}_\lambda\}$. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\inf_{\mathscr{S}_\lambda} \|v - v_g\|_{H^1} \leqslant \delta$, then the solutions of* (1) *corresponding to initial data v and v_g, denoted $v(z)$ and $v_g(z)$, satisfy $\sup_z \inf_{\mathscr{S}_\lambda} \|v(z) - v_g(z)\|_{H^1} \leqslant \varepsilon$.*

Proof. We argue by contradiction. Let $v_k(0)$ be a sequence of initial conditions such that $\inf_{\mathscr{S}_\lambda} \|v_k(0) - v_g\|_{H^1} \to 0$, and assume that $v_k(z)$ and $v_g(z)$ satisfy $\sup_z \inf_{\mathscr{S}_\lambda} \|v_k(z) - v_g(z)\|_{H^1} \geqslant \varepsilon$ for some $\varepsilon > 0$. For definiteness, let z_n to be the first time that $\inf_{\mathscr{S}_\lambda} \|v_k(z) - v_g(z)\|_{H^1} = \varepsilon$. By conservation of the L^2 norm and of the Hamiltonian, we have

$$\int_{\mathbb{R}} |v_k(z_n)|^2 \,\mathrm{d}t = \int_{\mathbb{R}} |v_k(0)|^2 \,\mathrm{d}t,$$

$$\mathscr{H}\{v_k(z_n)\} = \mathscr{H}\{v_k(0)\}.$$

By the assumption on $v_k(0)$ and continuity of $\mathscr{H}$, we have

$$\int_{\mathbb{R}} |v_k(z_n)|^2 \,\mathrm{d}t = \int_{\mathbb{R}} |v_k(0)|^2 \,\mathrm{d}t \to \lambda,$$

$$\mathscr{H}\{v_k(z_n)\} = \mathscr{H}\{v_k(0)\} \to \mathscr{H}\{v_g\},$$

i.e. $v_k(z_n)$ is nearly a minimizing sequence for $\mathscr{H}$. If we choose for example

$$w_k(z_n) = \frac{\lambda^{1/2} v_k(z_n)}{(\int_{\mathbb{R}} |v_k(z_n)|^2 \,\mathrm{d}t)^{1/2}},$$

then $w_k(z_n)$ is a sequence of functions in $\mathscr{A}_\lambda$ such that

$$\|w_k(z_n) - v_k(z_n)\|_{H^1} \to 0.$$

By continuity of $\mathscr{H}$, $w_k(z_n)$ is a minimizing sequence for $\mathscr{H}$ and must have a subsequence $w_{m_k}(z_n)$ which converges to a member of $\mathscr{S}_\lambda$. Now

$$\|v_k(z_n) - v_g(z)\|_{H^1} \leqslant \|v_k(z_n) - w_{m_k}(z_n)\|_{H^1} + \|w_{m_k}(z_n) - v_g(z)\|_{H^1}$$

and taking the infimum over $\mathscr{S}_\lambda$ gives

$$\begin{aligned}\varepsilon &= \inf_{\mathscr{S}_\lambda} \|v_k(z_n) - v_g(z)\|_{H^1} \\ &\leqslant \|v_k(z_n) - w_{m_k}(z_n)\|_{H^1} + \inf_{\mathscr{S}_\lambda} \|w_{m_k}(z_n) - v_g(z)\|_{H^1} \to 0,\end{aligned}$$

a contradiction. Thus the class of ground states must be orbitally stable. □

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Appendix A. Ground states for zero average dispersion

An interesting problem from both analytical and physical viewpoints [23,24] is the variational problem corresponding to zero average dispersion, namely
Minimize

$$\mathscr{H}(v) = -\frac{1}{2}\int_{\mathbb{R}} K(s) \int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^4 \,\mathrm{d}t\,\mathrm{d}s$$

over the set of admissible functions

$$\mathscr{A}_\lambda = \left\{ v \in L^2(\mathbb{R}), \int_{\mathbb{R}} |v|^2 = \lambda \right\}.$$

This problem was first analyzed successfully for the special case of K the indicator function on $[0, 1]$ and $\alpha_1 = 0$, $\alpha_3 = 0$ and subsequently for $\alpha_1 = 0$ [13,14]. Due to Strichartz' estimates [11], the corresponding functionals in these special cases are bounded from below in $\mathscr{A}_\lambda$, and it was shown that vanishing and splitting of the minimizing sequence is not possible in both Fourier and physical space. Since the problems are essentially localized in Fourier and in physical space, one can apply the classical Sobolev embedding theorems to show that the minimizing sequence converges strongly in $L^2(\mathbb{R})$ to a ground state.

In the general problem above, one can again use Strichartz' estimates to bound the Hamiltonian from below. We first set $a = |\mathscr{L}(s)\{v\}|^8$, $b = |\mathscr{L}(s)\{v\}|^2$ and $\theta = \frac{1}{3}$ in the elementary inequality

$$a^\theta b^{1-\theta} \leqslant \theta a + (1-\theta) b$$

which holds for $a, b \geqslant 0$ and $\theta \in (0, 1)$, and integrate over $\mathbb{R}$ to obtain

$$\begin{aligned}\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^4 \,\mathrm{d}t &\leqslant \frac{1}{3}\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^8 \,\mathrm{d}t + \frac{2}{3}\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^2 \,\mathrm{d}t \\ &= \frac{1}{3}\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^8 \,\mathrm{d}t + \frac{2}{3}\int_{\mathbb{R}} |v|^2 \,\mathrm{d}t.\end{aligned}$$

The appropriate Strichartz' estimate for the operator $\mathscr{L}$ is

$$\int_{\mathbb{R}^2} |\mathscr{L}(s)\{v\}|^8 \,\mathrm{d}t\,\mathrm{d}s \leqslant C\left(\int_{\mathbb{R}} |v|^2 \,\mathrm{d}t\right)^4 = C\lambda^4 < \infty$$

[11], so that $F(s) = \int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^8 \,\mathrm{d}t < \infty$ for almost every s. Overall,

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^4 \,\mathrm{d}t\,\mathrm{d}s \\
&\quad\geqslant -\frac{1}{2}\int_{\mathbb{R}} K(s)\left(\frac{1}{3}\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^8 \,\mathrm{d}t + \frac{2}{3}\int_{\mathbb{R}} |v|^2 \,\mathrm{d}t\right) \\
&\quad= -\frac{1}{6}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |\mathscr{L}(s)\{v\}|^8 \,\mathrm{d}t\,\mathrm{d}s - \frac{1}{3}\int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |v|^2 \,\mathrm{d}t\,\mathrm{d}s \\
&\quad= -\frac{1}{6}\int_{\mathbb{R}} K(s)F(s)\,\mathrm{d}s - \frac{\lambda}{3}\int_{\mathbb{R}} K(s)\,\mathrm{d}s \\
&\quad\geqslant -c_1\|F\|_{L^\infty(\mathbb{R})} - c_2\lambda > -\infty.
\end{aligned}$$

In principle, it should be possible to adapt the analysis in [14] to this problem to obtain a ground state. Also, the use of Strichartz' estimates above suggests that it may be possible to extend Theorem 1.1 for the case $K \in L^p(\mathbb{R})$ for some range of p with $p > 1$.

Appendix B. Numerical algorithm for calculation of the ground state

The ground state for the constrained minimization problem is a weak solution of the Euler–Lagrange equation

$$-\omega v + v_{tt} + \mathscr{N}\{v\} = 0 \tag{B.1}$$

or

$$-\omega\hat{v} - k^2\hat{v} + \hat{\mathscr{N}}\{v\} = 0 \tag{B.2}$$

in Fourier domain. The computation of solutions to this eigenvalue problem is of great interest and practical importance in the model's application to optical communications. The form of the nonlinearity makes it amenable to local computation, and does not require the evaluation of multiple integrals in Fourier domain. This technique was utilized in [16,17] to analyze the tails of the ground state in various parameter regimes, and was later adapted to compute ground states in the case of pinned random dispersion [9]. Here we present the algorithm for our general problem (B.2). It is based on the following series of iterations.

$$\hat{v}_{n+1} = \frac{\hat{\mathscr{N}}\{v_n\}}{\omega_n + k^2}, \tag{B.3}$$

$$\omega_{n+1} = \frac{\int_{\mathbb{R}} \bar{\hat{v}}_{n+1}(k)\hat{\mathscr{N}}\{v_{n+1}\}\,\mathrm{d}k - \int_{\mathbb{R}} k^2|\hat{v}_{n+1}(k)|^2\,\mathrm{d}k}{\int_{\mathbb{R}} |\hat{v}_{n+1}(k)|^2\,\mathrm{d}k}. \tag{B.4}$$

In practice, the algorithm is

1. Choose an initial profile $\hat{v}_0$, typically Gaussian.
2. Compute ω_0 using (B.4) with $\hat{v}_0$.
3. Compute $\hat{v}_1$ using (B.3)
4. Rescale $\hat{v}_1$ so that $\|\hat{v}_1\|^2_{L^2(\mathbb{R})} = \|\hat{v}_0\|^2_{L^2(\mathbb{R})}$. Call this $\hat{v}_1'$.
5. Compute ω_1 using (B.4) with $\hat{v}_1'$.
6. Repeat steps 3, 4, and 5 until the desired accuracy is reached.

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